Thermodynamics of critical strange nonchaotic attractors

Sandip Datta, Ashutosh Sharma, and Ramakrishna Ramaswamy
School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110 067, India
(Received 11 February 2003; revised manuscript received 5 May 2003; published 10 September 2003)

The thermodynamic formalism is applied to dynamical attractors which have fractal geometry and on which all Lyapunov exponents are exactly zero. Such *critical* strange nonchaotic attractors (SNAs) which arise, for example, in the Harper system exhibit a static phase transition in the free energy. The Tsallis nonextensive entropy which is known to characterize the thermodynamics of systems with leading Lyapunov exponent zero is found to be subadditive for the critical states. These properties are shared by other quasiperiodic systems with critical SNAs.

DOI: 10.1103/PhysRevE.68.036104 PACS number(s): 05.70.Ln, 05.45.Df, 05.90.+m

I. INTRODUCTION

Strange nonchaotic dynamics [1,2] is generic in quasiperiodically driven dynamical systems. Such motion, which is confined to attractors that have a fractal structure, is characterized by having a nonpositive largest Lyapunov exponent. Thus the motion is globally stable, although over short segments of a trajectory, the dynamics can be unstable, with positive local Lyapunov exponent [3].

If all the Lyapunov exponents are strictly zero, then such strange nonchaotic attractors (SNAs) are termed *critical*. The creation and stability of such attractors have been studied extensively in the past few years, mainly in a variety of dynamical systems of the form

$$x_{n+1} = -[x_n - E + V_n]^{-1}, \tag{1}$$

where V_n is a quasiperiodic function of the index n. Such a discrete mapping can be shown to obtain from the discrete Schrödinger equation

$$\psi_{n+1} + \psi_{n-1} + V_n \psi_n = E\psi_n, \tag{2}$$

where the potential V_n is a function of the lattice site n. The variable x_n in Eq. (1) is the ratio of wave-function amplitudes on neighboring sites, ψ_{n-1}/ψ_n.

For E an eigenvalue of Eq. (2), it can easily be seen that the inverse localization length is the Lyapunov exponent of the dynamics generated in the mapping, Eq. (1). Thus localized states necessarily correspond to attractors, while extended states, having infinite localization length, have Lyapunov exponent equal to zero.

When the potential $V_n = 2\epsilon\cos 2\pi(n\omega + \phi_0)$, Eq. (2) describes an electron moving in two dimensions on a square lattice in the presence of a transverse magnetic field [4]. In an appropriate gauge, the problem reduces to a one-dimensional eigenvalue equation, ω, ϵ, and ϕ_0 being parameters that determine the modulation, strength, and phase of the potential, respectively. Alternately, Eq. (2) can also be viewed as a tight binding model, and for this choice of potential it is often referred to as the Azbel-Hofstadter problem [5,6]. For the case of irrational ω when the potential is quasiperiodic, extensive studies, both numerical and analytic, have established that the Harper [4] (or the "almost-Mathieu" [7]) equation

$$\psi_{n+1} + \psi_{n-1} + 2\epsilon\cos 2\pi(n\omega + \phi_0)\psi_n = E\psi_n \tag{3}$$

exhibits a metal-insulator transition at $\epsilon = 1$, going from a phase where all states are extended to one where all states are exponentially localized [8]. At the critical value of $\epsilon = 1$, the eigenspectrum is singular continuous [9] and the wave functions are power-law localized [10] with fractal fluctuations [11]. For quasiperiodic potentials V_n generated by discrete quasiperiodic sequences, for example, the Fibonacci lattice [12] or the Kohmoto model [13], all eigenstates are also power-law localized [14].

Critical localization is thus ubiquitous in quasiperiodic systems. The Hofstadter butterfly [6], which is the spectrum of the Harper map at $\epsilon = 1$, viewed as a function of $\omega \in [0,1]$ thus consists of critical states for all irrational ω. Signatures of this spectrum have been seen in recent experiments [15], further underscoring the importance of such eigenstates.

When studied as orbits of the equivalent dynamical system, critical states must correspond to attractors with the largest Lyapunov exponent equal to zero. In earlier work, we have shown that these attractors are also SNAs, and in the present work we examine the properties of such critical SNAs.

We study orbits of the dynamical system, Eq. (1), corresponding to the eigenvalue equation, Eq. (3). The Harper map [16] which can be written explicitly as the following two-dimensional skew-product mapping

$$x_{n+1} = -[x_n - E + 2\epsilon\cos 2\pi\phi_n]^{-1}, \tag{4}$$

$$\phi_{n+1} = \omega + \phi_n \mod 1, \tag{5}$$

has been studied extensively [11,16–18] in order to examine the connection between strange nonchaotic dynamics [1] and localized states in quasiperiodic potentials [19,20]. (The subscript n is now the iteration index, and E, ϵ, and ω are parameters.)

We apply the thermodynamic formalism [21,22] to the study of attractors corresponding to such critical SNAs. From previous work it is known that the dynamics of the Harper map for E, an eigenvalue of the Harper equation, and $\epsilon = 1$ is on SNAs which have both Lyapunov exponents zero. Such attractors have an unusual symmetry of local Lyapunov

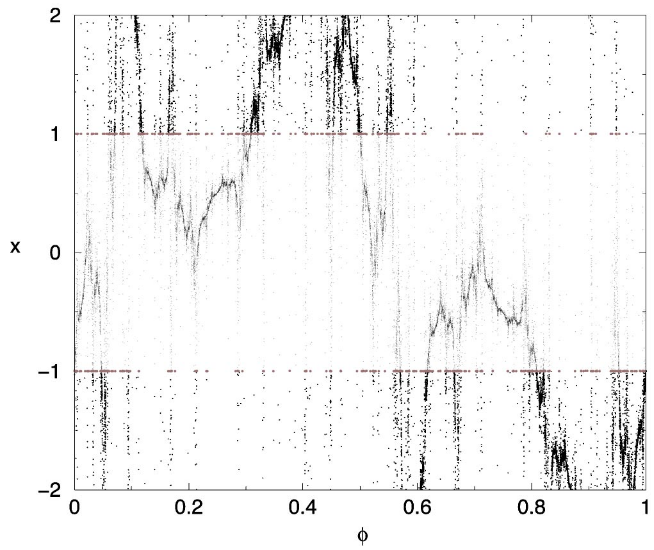

FIG. 1. Attractor of the Harper map at $\epsilon = 1.0, E = 0$. Regions where the stretch exponent is positive and negative are shown with big dots, and small dots, respectively. These regions are separated by the lines $x = \pm 1$.

exponents [17,23], a feature which is also shared by attractors of maps corresponding to critical states of a variety of quasiperiodic Schrödinger equations [18,23].

Although the critical attractor is nonchaotic, the dynamics can be locally unstable which is reflected in the fractal geometry of the SNA. The need to describe such states by a spectrum of singular measures leads us to characterize critical SNAs through their multifractal properties. Earlier studies on the multifractal properties of wave functions of localized states [10,24] have suggested that phase transition behavior may be linked to localization [25]. The present work reveals connections between critically localized systems and phase transitions in the multifractal spectrum.

II. ATTRACTORS OF THE HARPER MAP

For our numerical studies of the mapping, Eqs. (4) and (5), we fix the value of ω at $(\sqrt{5}-1)/2$, which is the inverse of the golden mean ratio. The Harper map is nonhyperbolic, and the critical attractors, an example of which is given in Fig. 1 for eigenvalue $E = 0$, have a highly nonuniform density. (Only when E is an eigenvalue does the dynamics occur on a SNA; if E is not an eigenvalue, the Lyapunov exponent is strictly negative and the dynamics is on a torus attractor [18].) The attractor shown in Fig. 1 is typical for most eigenvalues E.

The Harper map is also an example of a random Möbius transformation

$$x_{n+1} = \frac{a x_n + b}{c x_n + d}, \tag{6}$$

with $a = 0, b = -1, c = 1$, and with $d = V_n - E$ changing from iteration to iteration. The stability characteristics are easily ascertained: stable and unstable regions on this attractor are indicated (in black and gray, respectively) with the lines $x = \pm 1$ forming the boundary between them.

When all Lyapunov exponents are zero, the separation between trajectories increases only as a power in time rather than exponentially [17]. The dynamics on SNAs samples the stable and unstable regions in a complicated manner so that although the dynamics is nonchaotic in the sense of having a nonpositive global Lyapunov exponent, there is also intermittency, owing to local instability. There is thus the possibility of multifractal fluctuations in the critical attractor and we examine this below.

A. Multifractality

We apply the thermodynamic formalism [21] and find that critical SNAs in the Harper system exhibit the so-called static phase transition [26]. A ‘‘phase transition’’ in the context of thermodynamics applied to dynamical systems is signaled by singularities in the free energy at a critical parameter value. This phenomenon typically occurs in nonhyperbolic dynamical systems [27].

The static phase transition can be seen by examining the free energy

$$G(q) = \lim_{N\to\infty} \frac{1}{N} \ln Z_N(q), \tag{7}$$

where

$$Z_N(q) = \sum_{i=1}^{N} \ln p_i^q \tag{8}$$

is the partition function. This is obtained by coarse graining the attractor and partitioning it into N cells of side l. The probability density in the ith cell being denoted by p_i and q is a real parameter [28]. The multifractal nature of the attractor is directly verified through the spectrum of generalized dimensions which are defined by [29]

$$\tau_q \equiv (q-1) D_q = \lim_{l\to 0} \frac{\ln Z_N(q)}{\ln l}. \tag{9}$$

In Fig. 2, τ_q is shown as a function of q. The change in slope at $q = 0$ is equivalent to nonanalytic behavior in the free energy, and denotes a phase transition at this point [22]. The main implication of the static phase transition is that the scaling behavior in the probability density of the attractor is not uniform, and this is probed by varying q.

Because of the skew-product nature of the Harper map, the attractor density is uniform in ϕ since this latter dynamics is merely a rigid rotation. For the x subsystem, the Frobenius-Perron operator can be written as

$$L\rho(x') = \sum \frac{\rho(x)}{|\mathcal{J}(x)|}, \tag{10}$$

where the sum is over all preimages x of x' under the map and $\mathcal{J} = |\partial x_{n+1}/\partial x_n| = x_{n+1}^2$ is the Jacobi determinant of the map. Points in the stable region in Fig. 1, which shows a

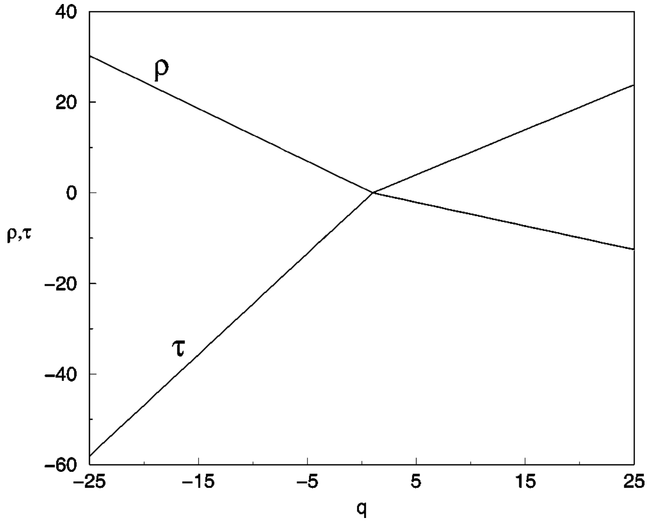

FIG. 2. Behavior of τ_q and $\rho_q=(q-1)\mathcal{E}_q$ as a function of q.

typical critical SNA, show that the bulk of the density resides in the stable region, the strip $|x|<1$, and these points typically have preimages in the unstable region $|x|>1$ which is naturally much less dense. This contributes to the origin of nontrivial scaling behavior, even though the dynamical system is reversible and nonchaotic.

Such behavior is typical near homoclinic tangencies. At such points the map is nonhyperbolic and the invariant density scales nontrivially. So with variation of q, the dominating contribution to the partition function comes either from the cells containing the homoclinic tangencies or from the rest of the attractor. The separation of the attractor into set of points with two different phases is manifested in the static phase transition [27]. The creation of SNAs in the Harper system through homoclinic collisions [17] may be one reason for the existence of such a phase transition.

The standard multifractal treatment [30] as applied above uses the gauge function with power law dependence on lengths, given by $\Lambda_q(l)=l^{-\tau}$. If the measure on the attractor is composed of subsets of power-law singularities of strength α [31], then α is related to τ and q through the relation $\alpha=d\tau/dq$. The static phase transition indicates that probability p_i in certain cells in the phase space might scale differently from l^{α}, which is the scaling form assumed in the choice of gauge function. In order to be able to employ a more appropriate choice of gauge function we note that for the Harper system, fluctuations in supercritical states can be examined by factoring out an exponentially decaying part [11]. The wave function at site k is then written as $\psi_k=\exp(-k\gamma)\eta_k$, where γ is the inverse localization length. Given the correspondence between critical SNAs and critical states, it is suggestive that the measure on the attractor can be expressed in terms of a generalized gauge function [29]

$$\Lambda_q(l)=[\exp(1/l)^R]^{\rho_q}l^{-\tau_q}, \tag{11}$$

using which the partition function Γ is defined through the cell probabilities p_i as

$$\Gamma(q)=\lim_{l\to 0}\sum_i p_i^q\Lambda_q(l_i). \tag{12}$$

When Γ is finite and nonzero we have $\rho_q=(q-1)\mathcal{E}_q$, where $\mathcal{E}_q$ is the spectrum of *exponential* dimensions [29]. Introducing the gauge function as

$$\lambda_q(l_i)=\prod_{j=1}^{n_1}[\exp^{(j)}(1/l_j)^{R_j}]^{\rho_j}l_i^{-\tau}, \tag{13}$$

where $\exp^{(j)}(x)=\exp[\exp^{(j-1)}(x)]$, $\exp^{(0)}(x)=x$. The partition function for finite l can be written as

$$\Gamma(q,\tau,\rho,R,l)=\sum_j N_jp_j^q\exp[\rho(1/l_j)^R]l_j^{\tau}. \tag{14}$$

The power R in Eq. (11) comes from the form of harmonic length scale dependence present in the system and its value is determined from the condition that at $q=0,\mathcal{E}_q=D_q$ [29]. In the present case we find the value of R numerically to be 0.418 93, and N_j is the number of cells of size l_j with corresponding probability p_j. Taking the limit $l\to 0$ so as to keep Γ nonzero and finite, we obtain the expression for exponential dimensions as

$$(q-1)\mathcal{E}_q=-\lim_{l\to 0}\max[l_j^R\ln(N_jp_j^q)]. \tag{15}$$

The static phase transition is evident at $q=0$ in the behavior of $\mathcal{E}_q$, which is shown in Fig. 2.

Note that in the Harper system, the stable region $|x|<1$ occupies a vanishing fraction of the entire phase space. The attractor, which has capacity dimension 2, has portions in both the stable and the unstable regions. Owing to the symmetry of the stretch exponents, the integrated density in these two regions are equal, but cells in the unstable regions necessarily have lower density. These dominate the contribution for $q<0$: here the corrections to power-law scaling are dominant, making $\mathcal{E}_q$ the leading dimension. Above $q=0$, the high density region is probed; these are mainly in the stable regions, where the pure power-law behavior takes over, and D_q is the leading dimension.

We have studied other critical SNAs in the Harper system (as well as in other maps) and find that this feature is characteristic of all such critical SNAs [32].

B. Nonextensivity

Sensitivity to initial conditions on an attractor is measured by the asymptotic Lyapunov exponent. When this exponent is zero, the system is "at the edge of chaos" or marginal [33], and requires more detailed examination [34]. A number of quantities show a weaker sensitivity, diverging only as a power of the control parameter. On critical SNAs, for instance, the Lyapunov exponent converges to zero as a power [17].

Generalizations of the Boltzmann-Gibbs entropy, namely, $S=-\Sigma_i p_i\ln p_i$, to reflect this weaker convergence have been suggested [34,35], under the framework of the so-called nonextensive thermodynamics. From the scaling properties of multifractals the following form of the entropy [35] was proposed:

$$S_Q = \frac{1}{(1-Q)}\left\{1-\sum_{i=1}^{m} p_i^Q\right\}, \tag{16}$$

which in contrast to the usual Boltzmann-Gibbs form of the entropy (to which it also reduces in the limit $Q \to 1$) is nonextensive [35] for $Q \neq 1$. The Tsallis entropic index Q^* is the appropriate value of Q for the map under consideration, and the corresponding system is subadditive if $Q^* < 1$ or superadditive if $Q^* > 1$. The Tsallis formalism has been applied largely for one-dimensional maps in the marginal case, namely, for generalized logistic maps [33] and circle maps [36]. These studies reveal that the power-law sensitivity to initial condition is more suitably described in terms of a Q^* value different from unity, indicating nonextensive nature of the thermostatistics at the period doubling and tangent bifurcation points and also near onset of chaos. Critical SNAs, with all Lyapunov exponents zero and an underlying fractal structure, are clearly suitable candidates on which the Tsallis formalism can be applied.

We compute the Tsallis entropic index Q^* from the nonextensive entropy following the procedure suggested by Tsallis and co-workers [37]. The phase space is divided into m equal cells. A set of N initial points are selected in one of these. Under the dynamics, these points spread to other cells, and we follow the quantities $p_i(t)$, namely, the occupation probability of cell i at time t. Using Eq. (16) we compute the Tsallis entropy as a function of time.

When $t=0$, probabilities of all the boxes are zero except one, and hence $S_Q(t)=0$. $S_Q(t)$ increases with time, but its growth is bounded from above by a constant saturation value corresponding to the equilibrium distribution of points on the attractor. Therefore $S_Q(t)$ saturates in the long time limit for all values of Q. Prior to saturation, and after an initial transient period, $S_Q(t)$ varies linearly only for $Q=Q^*$, thereby defining the Tsallis entropic index.

For the critical SNA at band center, we find that $Q^* \approx 0.82$. For $Q=Q^*$, $S_Q(t)/t$ remains finite, whereas for $Q < Q^*$, $S_Q(t)/t$ diverges while for $Q > Q^*$, $S_Q(t)/t \to 0$. When the system shows strong sensitivity to initial conditions, which is characterized by positive Lyapunov exponent, Q^* is 1. But for systems at the edge of chaos with leading Lyapunov exponent being zero, the system shows weak sensitivity to initial conditions, and the value of Q^* is expected to be different from and smaller than 1. The value of Q^* gives a measure of the time required for the initial conditions to spread and reach the equilibrium distribution. So the time when the system begins to saturate depends on the value of Q^*. In Fig. 3, such tendency towards saturation becomes evident at around $t=38$.

As suggested by Latora *et al.* [37], a sensitive method for identifying linear growth in $S_Q(t)$ is to fit the curves to a quadratic $a+bt+ct^2$ in the interval $[t_1, t_2]$ and minimize the nonlinear term. Defining $r=|c|(t_1+t_2)/b$ [37], note that $r=0$ implies ideal linearity. The times t_1 and t_2 correspond

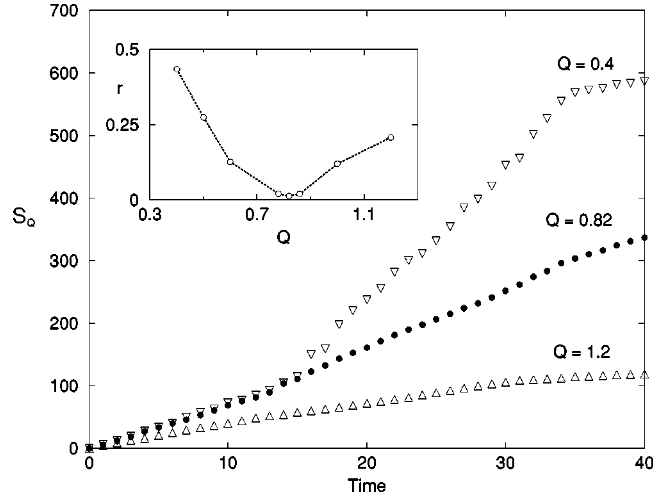

FIG. 3. Variation of S_Q with iteration for different Q. At $Q = Q^* \approx 0.82$ S_Q increases linearly. The cutoff n_t is taken to be 7000, and the resulting curve is an average over 942 runs. The variation of nonlinearity index r with Q is shown in the inset. For calculation of r, we take $t_1=12$ and $t_2=35$ for each value of Q. At later times S_Q approaches a constant value.

to the end of initial transient period and the onset of saturation, respectively. The variation of r with Q (inset of Fig. 3) for the present case shows that S_Q does indeed have a linear growth for $Q=Q^*$. We choose $t_2=35$ for the calculation of r for all Q; initial transients decay by $t=10$, and therefore an appropriate value for $t_1=12$.

Efficient averaging over the initial conditions is required to minimize numerical fluctuations [38]. We partition the phase space in a set of m equal sized cells. A number N initial conditions (x,ϕ) are selected inside a given cell i and allowed to evolve under the map dynamics for a specified time. We determine, at each time step, the *cumulative* number of cells that are occupied and denote this number as n_i. The procedure is repeated for each of the cells. For averaging, we include only those cells that have maximal spreading, namely, those initial cells for which n_i exceeds a threshold n_t.

This method does not give the value of entropic index Q^* to any higher precision due to numerical fluctuation, but nevertheless confirms the nonextensive nature of the attractor at the critical point of the Harper map. Furthermore, $Q^* < 1$ indicating subadditive thermostatistics. In case of power-law mixing of the phase space, as in the case of the critical attractor studied here, the subadditive nature of the entropy is expected [36]. The weak sensitivity of the initial condition also implies that the geometrical support of the attractor is multifractal [37]. The Tsallis entropy [35] is related to Renyi entropies, and thus signatures of this behavior can be expected in the multifractal and dimensional spectra of the map. In a related context [25], subadditivity has also been conjectured to be equivalent to localization. The present finding lends support to this conjecture.

III. SUMMARY

It is known that fluctuations in wave functions in the Harper system are self-similar and universal [11]. Our present results show that there are higher-order effects that more properly describe critical states. Although critical wave functions are localized, the envelope of the probability density shows very slow decay, with regions of slow variation as well as regions of relatively more rapid variation. The fluctuations in these different types of regions have separate and distinct behaviors: they are effectively probed by the multifractal dimensions D_q and $\mathcal{E}_q$, respectively.

The nonextensive thermodynamics [34] has been shown to apply to a variety of dynamical systems [35] with leading Lyapunov exponent zero. Critical SNAs in the Harper system are therefore natural candidates for application of the Tsallis formalism. While we find that these systems have a subadditive thermodynamics, it should be pointed out that unlike the typical cases that have hitherto been studied (such as the logistic map at the period-doubling accumulation point [36]) where small perturbations throw them into a chaotic phase, the dynamics in the Harper map (or maps derived from substitutional sequences such as Fibonacci map [39]) is always nonchaotic. Thus although the attractors are fractal, there can never be dynamical instability: Lyapunov exponents are strictly nonpositive. The present findings are quite general and apply not only to the critical SNAs in the Harper system but also to the critical SNAs in other systems (such as the Fibonacci or the Thue-Morse chain) as well [32].

Recent experiments [15] have realized the Harper system: evidence of the fractal energy spectrum for irrational ω has been seen in quantized Hall measurements. Since these states are all examples of critical localization, our present results show that this provides a suitable and robust experimental system in which the implications of nonextensive thermodynamics can be studied.

ACKNOWLEDGMENT

This work was supported by a grant from the Department of Science and Technology.

[1] A. Prasad, S.S. Negi, and R. Ramaswamy, Int. J. Bifurcation Chaos Appl. Sci. Eng. **11**, 291 (2001).

[2] C. Grebogi, E. Ott, S. Pelikan, and J. Yorke, Physica D **13**, 261 (1984).

[3] A.S. Pikovsky and U. Feudel, Chaos **5**, 253 (1995).

[4] P.G. Harper, Proc. Phys. Soc., London, Sect. A **68**, 874 (1955).

[5] M.Ya. Azbel, Sov. Phys. JETP **19**, 634 (1964).

[6] D. Hofstadter, Phys. Rev. B **14**, 2239 (1976).

[7] Y. Last, in *Proceedings of the XI International Congress of Mathematical Physics*, Paris, 1994, edited by D. Iagolnitzer (International Press, Cambridge, MA, 1995), pp. 366–372; S. Jitomirskaya, *ibid.*, pp. 373-382; A. Gordon, S. Jitomirskaya, Y. Last, and B. Simon, Acta Math. **178**, 169 (1997).

[8] G. André and S. Aubry, Ann. Isr. Phys. Soc. **3**, 133 (1980).

[9] J. Bellisard, R. Lima, and D. Testard, Commun. Math. Phys. **88**, 107 (1983).

[10] M. Kohmoto, Phys. Rev. Lett. **51**, 1198 (1983); M. Kohmoto, L.P. Kadanoff, and C. Tang, *ibid.* **50**, 1870 (1983); H. Hiramoto and M. Kohmoto, Int. J. Mod. Phys. B **6**, 281 (1992).

[11] J.A. Ketoja and I.I. Satija, Phys. Rev. Lett. **75**, 2762 (1995).

[12] Q. Niu and F. Nori, Phys. Rev. Lett. **57**, 2057 (1986); F. Piéchon, M. Benakli, and A. Jagannathan, *ibid.* **74**, 5248 (1995).

[13] M. Kohmoto, Phys. Rev. Lett. **51**, 1198 (1983).

[14] R. Ketzmerick, K. Kruse, S. Kraut, and T. Geisel, Phys. Rev. Lett. **79**, 1959 (1997).

[15] C. Albrecht, J.H. Smet, K. von Klitzing, D. Weiss, V. Umansky, and H. Schweizer, Phys. Rev. Lett. **86**, 147 (2001).

[16] J.A. Ketoja and I.I. Satija, Physica D **109**, 70 (1997).

[17] A. Prasad, R. Ramaswamy, I.I. Satija, and N. Shah, Phys. Rev. Lett. **83**, 4530 (1999).

[18] S.S. Negi and R. Ramaswamy, Phys. Rev. E **64**, 045204 (2001).

[19] See, e.g., J.B. Sokoloff, Phys. Rep. **126**, 189 (1985); M.Y. Azbel, P. Bak, and P.M. Chaikin, Phys. Rev. Lett. **59**, 926 (1987); I.I. Satija, Phys. Rev. B **49**, 3391 (1994).

[20] T. Geisel, R. Ketzmerick, and G. Petschel, Phys. Rev. Lett. **66**, 1651 (1991); in *Quantum Chaos*, edited by G. Casati and B. Chirikov (Cambridge University Press, Cambridge, 1995), pp. 633–60.

[21] D. Ruelle, *Thermodynamic Formalism* (Addison-Wesley, Reading, MA, 1978).

[22] C. Beck and F. Schlögl, *Thermodynamics of Chaotic Systems* (Cambridge University Press, Cambridge, 1993); A. Csordas and P. Szepfalusy, Phys. Rev. A **39**, 4767 (1989).

[23] R. Ramaswamy, Eur. Phys. J. B **29**, 339 (2002).

[24] L. Pietronero, A.P. Siebesma, E. Tostati, and M. Zannetti, Phys. Rev. B **36**, 5635 (1987).

[25] P. Grigolini, M.G. Pala, L. Palatella, and R. Roncaglia, Phys. Rev. E **62**, 3429 (2000); see also L. Palatella and P. Grigolini, e-print quant-ph/9810083.

[26] E. Ott, W. Withers, and J.A. Yorke, J. Stat. Phys. **36**, 697 (1984); P. Szepfalusy and T. Tel, Phys. Rev. A **34**, 2520 (1986); T. Bohr and M.H. Jensen, *ibid.* **36**, 4904 (1987); D. Katzen and I. Procaccia, Phys. Rev. Lett. **58**, 1169 (1987).

[27] P. Grassberger, R. Badii, and A. Politi, J. Stat. Phys. **51**, 135 (1988).

[28] The number of cells used for coarse graining the attractor is $36\,000\times 360$ and 10^7 points on the attractor are used to estimate the probabilities.

[29] R.E. Amritkar, A.D. Gangal, and N. Gupte, Phys. Rev. A **36**, 2850 (1987); N. Gupte and R.E. Amritkar, *ibid.* **41**, 4285 (1990).

[30] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, and B.I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[31] S.K. Sarkar, Phys. Rev. A **36**, 4104 (1987).

[32] Studies on other critical SNAs in Harper and other systems can be found in S. Datta (unpublished).

[33] U. Tirnakli, G.F.J. Ananos, and C. Tsallis, Phys. Lett. A **289**, 51 (2001).

[34] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).

[35] *Nonextensive Statistical Mechanics and its Applications*, edited

by S. Abe and Y. Okamoto, Lecture Notes in Physics (Springer-Verlag, Berlin, 2001), and references therein.
[36] M.L. Lyra and C. Tsallis, Phys. Rev. Lett. **80**, 53 (1998).
[37] V. Latora, M. Baranger, A. Rapisarda, and C. Tsallis, Phys. Lett. A **273**, 97 (2000).
[38] The attractor is divided into 64×10^4 cells and in one of the cells, we take $N=64\,000$ initial conditions, chosen at random.
[39] S.S. Negi and R. Ramaswamy, Pramana, J. Phys. **56**, 47 (2001).